SINGLE PARTICLES AND COMPOSITE SYSTEMS IN A MATHEMATICALLY RIGOROUS FORMULATION OF RELATIVISTIC LAGRANGIAN QUANTUM FIELD THEORY

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Abstract

We define quantum field theory by taking the Lagrangian action to be given as a sequence of mathematically well-defined functionals written in terms of operator fields fulfilling given local commutation relations. The renormalized solution fields have a fully defined Fock space expansion and are multi-local; thus Haag's theorem does not apply, i.e., the interaction picture exists. Also, the formalism allows immediately the definition of a wave function and the description of many-body bound-state systems.

1 INTRODUCTION

In the present paper we will show that it is possible to formulate relativistic quantum field theory (QFT) in a mathematically rigorous way in that every step in the chain of reasoning, beginning with the Lagrangian and terminating with the S-matrix can be rigorously defined and that all the quantities participating in this chain exist as fully specified mathematical objects. In this formulation the interacting theory joins smoothly to the non-interacting case. At the same time the treatment of bound composite system presents no difficulties; in particular, it allows the definition of a system wave function which is fully analogous to the wave function of non-relativistic quantum mechanics. In detail, the modifications from the conventional formulation of QFT are: (i) the basic fields have to be defined as local quantum fields; (ii) the Lagrangian functional must be replaced by a generalized functional (short: g-functional), which is the analogue of the replacement of functions by generalized functions (short: g-functions), see below and Appendix A.

Of the above mentioned two modifications, only the first is new; the second has been implied but only rarely used explicitly. Conventionally it is postulated that the "renormalized" fields are local fields. In the present formulation the basic fields of the Lagrangian, i.e., the so-called "free" fields, are defined to be local fields, and to obey once and for all fixed commutation relations. These fields are used to construct a Fock space, and all quantities of the theory are expanded in that basis. This way the "renormalized" fields, i.e., the solution fields, as the result

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of the interactions acquire structure, i.e., they are non-local, and are not unitarily related to the "free" fields. Hence Haag's theorem [1] does not apply in our formulation, i.e., the interaction picture exists. This way it is this first of the modifications which is the basis for the resolution of the contradictions inherent in the conventional formulation. One may say that the present re-formulation entails the reversal of the roles of the "bare" and the "dressed" fields.

On the other hand, something along the lines of the second of the modifications, the introduction of the g-functional to achieve convergent expressions for the Feynman graphs, has been *implicitly* utilized previously. The reason for doing the calculation *explicitly* in terms of a g-functional rather than to regularize the divergences is to maintain control over the treatment so as to keep intact Noether's theorem and the other quantities and interrelations of the Lagrange-Hamilton based quantum theory.

It will be demonstrated that in the present formulation the Lagrangian g-functional can be chosen such that without any regularization:

(a) the Fock space is well-defined;

(b) the propagators are of the Pauli-Villars form and are the Green's functions for the Lagrangian g-functional constructed on the Fock space ;

(c) the solution state vectors are given by a normalizable expansion on the Fock space basis;

(d) the renormalization constants, as all the other quantities of the theory, are given unambiguously by absolutely convergent Feynman – Schwinger integrals;

(e) the solution fields, i.e., the fields which in the Heisenberg picture fulfill the equations of motion including the interactions, are non-local;

(f) Haag's theorem [1, 2, 3] does not apply [4];

(g) in the limit $\lambda \to 0$ (see below, Eq. (1.1) and the discussion) the renormalized S-matrix coincides with that computed in the conventional way;

(h) the theory obeys all Ward-Takahashi identities derivable from Noether's theorem [5, 6, 7, 8, 9];

(i) the concept of a wave function remains valid for relativistic QFT, and allows the description of many-body bound states;

(j) the solutions are analytic functions of the coupling constant, say g^2, as it tends towards zero;

(k) except for locality the theory fulfills the axioms of constructive field theory [10, 11], including that the physical quantities arising in QFT are at most tempered g-functions; and

(l) Källen's theorem [12] applies.

As is well known, the mathematical problems of the conventional formulation of QFT are the following:

(A) The starting point are the solutions of the $\lambda = 0$ form (in contrast to the $\lambda \to 0$ limit) of the g-functional of our formulation. Most likely such solutions do not exist for non-trivial theories. Symptomatic for this are, for example, the statements: the S-matrix has an essential singularity at $g^2 = 0$; Fock space does not exist for non-trivial QFT.

(B) In the regularization the Green's functions get mutilated by some prescription, be it the Pauli-Villars regularization or the dimensional regularization. Neither the regularized Green's functions, nor the remaining (finite) parts used in the calculation are the Green's functions of the problem at hand; they are disconnected from the assumed Lagrangian.

(C) The Green's functions of the theory break the boundaries of tempered g-functions; as has been demonstrated earlier [7, 8] they contain meta-g-functions (see Appendix A).

(D) According to the postulated properties of the (renormalized) fields, in particular their local character, Haag's theorem does apply and renders the results of QFT meaningless.

(E) The problem of the convergence of the perturbation expansion of the S-matrix for a generic (3+1) QFT, the empirically observed at least semi-convergence of QED notwithstanding, remains unsolved and is not addressed in the present paper.

In view in particular of the points (c) and (k) the mathematics of the present formulation is very much more simple than that of the conventional formulation.

We now give a short overview of the points which will be demonstrated in the paper.

The generalized (operator) functional, the g-functional, is defined by a sequence of functionals (see Appendix A):

$$L\{\psi_i, \ldots; \lambda_n\} \;=\; L_0\{\psi_i, \ldots; \lambda_n\} \;+\; L_1\{\psi_i, \ldots; \lambda_n\} \quad , \tag{1.1}$$

where λ_n is a parameter (e.g., having the dimension of a length). The functional is constructed such that for $n \to \infty$, i.e., for $\lambda_n \to 0$, $L\{\lambda_n\}$ approaches the desired Lagrangian action (say, the quantum electrodynamics (QED) Lagrangian action), and that for $\lambda \neq 0$ all Feynman integrals of the Schwinger — Tomonaga expansion of the S-matrix converge. The "basic fields" ψ_i etc., are quantized fields, i.e., operator fields, defined to obey given local commutation relations. They

can be chosen to fulfill the equations of motion resulting from $L_0\{\cdot\}$. The (trivial) question of normalizability of the continuum can be taken care of in any way one likes, e.g., by the Weyl eigendifferential method, or by periodic boundary conditions. The basic fields can be used to construct a Fock space. No difficulty is encountered in the expansion of any quantities of the full theory in this Fock basis since, in view of the convergence of all graphs, all such quantities are mathematically fully defined directly in the Fock space without any mutilation by cut-offs or regularization of any integrals. The solutions, e.g., the expansion of the state vectors, can be obtained either by perturbative methods, or non-perturbatively, by matrix diagonalization. This way no difficulties arise in the description of bound states of composite systems, like, for example, positronium.

In the present paper we provide a complete chain of the steps needed for the demonstration (of the mathematical aspects) of our formulation. However, we shall not give any proofs in terms of the rigorous mathematical language; our proofs, if at all given, will be in the simple language of a physicist. In fact, since the mathematical concepts arising in our formulation are rather simple, i.e., are at most tempered distributions (tempered g-functions), the need for the mathematician's language anyway is not that great.

For concreteness we conduct the discussion in terms of the simplest possible non-trivial cases, i.e., a g-functional which has as its limiting form a spinor field interacting with a pseudoscalar field (Section 2), and the case of spinor electrodynamics (Section 3). The respective g-functionals are written in terms of Heisenberg-picture quantized fields. They are manifestly relativistically covariant in general, i.e., not only in the limit $\lambda_n \to 0$. To continue the development we need to derive the Hamiltonian, i.e., we must apply the variational calculus which in our case we need for quantized rather than for classical fields. As shown in Ref. [13] the functional derivative, and hence the variational calculus, can be rigorously defined also for quantized fields, i.e., for operator fields and that then it yields the same expressions as for c-number fields. In view of this result, the conventional procedure which consists in splitting the Lagrangian into $L_0^{(n)}$ and $L_1^{(n)}$, the "free" and the "interactions" part; in deriving the corresponding Hamiltonian; and in going over to the interaction picture, is mathematically fully defined and can be applied. At this point one can, as done conventionally, compute the S-matrix. All these steps are performed using only the basic fields. *The solution fields, which are supposed to fulfill the complete equations of motion, are not required for computing the S-matrix.*

At this point one may terminate the development, as all perturbative predictions of the theory can now be computed. However, the physics and the need in particular for our modification (i) has remained hidden. This need becomes evident in the description of the solutions, in particular of bound many-body systems, as, for example, positronium. Thus, to demonstrate that the

modification (i) actually is basic to the viability of QFT, we continue the development, and we derive in Section 4 the form of the field operators, say Ψ, and of the state vectors, say $|W\rangle$, of a solution. In the g-functional formulation they are mathematically well-defined quantities. In particular, the solution fields can be expanded in terms of products of the basic fields; this demonstrates the existence of the Fock space for the interacting theory. Furthermore, this shows that the solution fields and the basic fields here are not unitarily equivalent; because of this Haag's theorem does not apply. The state vectors for any arbitrary physical system are normalizable; so are its wave functions, defined as usually as the matrix element $\langle 0|\ \Psi\ |W\rangle$; this demonstrates the existence of the wave function in relativistic QFT. As in the case of the Pauli-Villars regularization procedure, for low energy processes the auxiliary (ghost, i.e., negative metric and/or "wrong" statistics, and normal) particle degrees of freedom for $\lambda_n \to 0$ freeze out, except for providing for the convergence of all Feynman integrals. Owing to this convergence Källen's theorem [12] applies, which means that the $\lambda \ \to \ 0$ results are independent of the specific choice of the g-functional.

As for previous work, as already mentioned above, the ingredient (ii) of the present approach, i.e., the use of modified Lagrangians, both polynomial and non-polynomial, to achieve convergent Feynman integrals is very old [14, 15, 16, 17]. It has been employed in many papers; for example, in more recent work devoted to the proof of the existence of the ϕ^4 theory [18, 19] the development is carried out using regularized propagators without actually specifying the underlying Lagrangian; only the existence of such a Lagrangian is postulated.

A discussion of the formulation of QFT in terms of path integrals lies outside of the frame of this paper.

2 YUKAWA QUANTUM FIELD THEORY

The simplest possible non-trivial quantum field theory is provided by interacting massive spin 0 and spin $\frac{1}{2}$ fields. This system has no further symmetries beyond those imposed by Lorentz invariance. The presence of further invariances, e.g., gauge invariance, requires imposing constraints on the solutions. We shall treat such a case in the next Section. Here we consider the minimal theory, which, however, involves the full mathematical apparatus needed for the present re-formulation of QFT.

Consider the Lagrangian action g-functional

$$L\{\psi \ldots\} \ = \ L_F(\lambda) + L_B(\lambda) + L_I(\lambda) \quad , \tag{2.1}$$

(for simplicity we write λ instead of λ_n; also the limit $\lambda \to 0$ is understood throughout)

$$L_F(\lambda) \;=\; -\int d^4x \;:\; \{\xi_1\,\bar\psi_1(x)\;(\gamma\partial + m_1)\;\psi_1(x) \;+\; \xi_2\,\bar\psi_2(x)\;(\gamma\partial + m_2)\;\psi_2(x)$$

$$+\;\xi_3\,\bar\psi_3(x)\;(\gamma\partial + m_3)\;\psi_3(x)\}\;: \tag{2.2}$$

$$L_B(\lambda) \;=\; -\int d^4x \;:\; \left\{\tfrac{1}{2}\;(\partial_\mu(x))^2 \;+\; \tfrac{1}{2}\;(M\varphi(x))^2\right\}: \tag{2.3}$$

$$L_I \;=\; ig\int d^4x \sum_{i,j}\kappa_i\kappa_j \;:\; \bar\psi_i(x)\;\gamma_5\;\psi_j(x)\;\varphi(x)\,: \quad . \tag{2.4}$$

This form is manifestly relativistically invariant. It contains 2 "auxiliary" Fermion fields. As we will see, this is a possible choice; there exist infinitely many other possible choices.

The fields $\psi_i(x),\ \varphi(x)$ are defined to obey the commutation relations ($|\xi_i| \;=\; |\epsilon| \;=\; 1$, see below)

$$\left[\psi_{i\alpha}(\mathbf{x},t),\ \psi^\dagger_{j\beta}(\mathbf{y},t) \;+\; \epsilon\;\psi^\dagger_{j\beta}(\mathbf{y},t)\;\psi_{i\alpha}(\mathbf{x},t)\right] \;=\; \xi_i\,\delta_{ij}\,\delta_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \tag{2.5}$$

$$[\varphi(\mathbf{x},t),\ \dot\varphi(\mathbf{y},t)]_- \;=\; i\,\delta^3(\mathbf{x}-\mathbf{y}) \quad , \tag{2.6}$$

where α, β are spinor indices. Fields which have $\xi = -1$ are ghost fields while normal fields have $\xi = +1$, and $\epsilon = -1$ yields the "wrong" statistics. Further, we define

$$m_1 \;=\; m \tag{2.7a}$$

$$m_2 \;=\; \frac{c_2}{\lambda} \tag{2.7b}$$

$$m_3 \;=\; \frac{c_3}{\lambda} \tag{2.7c}$$

with c_2 and c_3 real positive constants. This defines the $\lambda \to 0$ limit of the g-functional. The c-number constants κ_i will be determined below.

Since, as shown in [13], variational calculus is valid not only for c-number but also for operator fields one can go through the usual procedure, i.e., use the non-interacting part of L to obtain H_0, go over to the interaction picture, and compute the S-matrix in the standard manner (see Appendix B). To that end one needs the graph expansion of the Tomonaga – Schwinger equation. In view of the form of (2.1), (2.2), this expansion will turn out to be exactly of the Pauli-Villars form as we now demonstrate.

Consider the Fermion Green's function as it arises, for example, in the Wick expansion of the second order term of the Neumann series of the Tomonaga – Schwinger equation:

$$G_F \;=\; T\sum_{jk}\kappa_j\kappa_k\;\langle 0|\;\psi_j(x)\;\bar\psi_k(y)\;|0\rangle \quad . \tag{2.8}$$

In view of the commutation relations (2.5) we have in momentum space (for brevity we omit the infinitesimal imaginary parts in the denominators)

$$\begin{aligned}
\widetilde{G}_F &= K_1\,\frac{\gamma p + m_1}{p^2 - m_1^2} \;+\; K_2\,\frac{\gamma p + m_2}{p^2 - m_2^2} \;+\; K_3\,\frac{\gamma p + m_3}{p^2 - m_3^2} \\
&= \Big[\gamma p\,\Big\{(K_1 + K_2 + K_3)\,p^4 \;-\; \Big[K_1(m_2^2 + m_3^2) \;+\; K_2(m_1^2 + m_3^2) \;+\; K_3(m_1^2 + m_2^2)\Big]\,p^2 \\
&\qquad + \Big[K_1 m_2^2 m_3^2 \;+\; K_2 m_1^2 m_3^2 \;+\; K_3 m_1^2 m_2^2\Big]\Big\} \\
&\qquad + \Big\{(K_1 m_1 + K_2 m_2 + K_3 m_3)\,p^4 \\
&\qquad + \Big[K_1 m_1(m_2^2 + m_3^2) \;+\; K_2 m_2(m_1^2 + m_3^2) \;+\; K_3 m_3(m_1^2 + m_2^2)\Big]\,p^2 \\
&\qquad + \Big[K_1 m_1 m_2^2 m_3^2 \;+\; K_2 m_2 m_1^2 m_3^2 \;+\; K_3 m_3 m_1^2 m_2^2\Big]\Big\}\Big] \\
&\times \Big[(p^2 - m_1^2)(p^2 - m_2^2)(p^2 - m_3^2)\Big]^{-1} \quad . \qquad (2.9)
\end{aligned}$$

Here

$$K_i \;=\; \xi_i\,\kappa_i^2 \quad . \qquad (2.10)$$

In order to cancel the terms of the numerator containing the factors $(\gamma p)\,p^4$ and p^4 there must hold

$$\sum K_i \;=\; 0 \qquad (2.11)$$

$$\sum m_i\,K_i \;=\; 0 \quad . \qquad (2.12)$$

Hence (we put $K_1 \;=\; 1$ so as to achieve the needed form for $\lambda \to 0$)

$$K_2 \;=\; -\;\frac{c_3 - \lambda m_1}{c_3 - c_2} \qquad (2.13a)$$

$$K_3 \;=\; \frac{c_2 - \lambda m_1}{c_3 - c_2} \quad . \qquad (2.13b)$$

With these relations (2.9) can be simplified to the expression

$$\begin{aligned}
\widetilde{G}_F &= \Big[\gamma p\,\Big\{\Big[K_1\,m_1^2 \;+\; K_2\,m_2^2 \;+\; K_3\,m_3^2\Big]\,p^2 \;+\; \Big[\frac{K_1}{m_1^2} \;+\; \frac{K_2}{m_2^2} \;+\; \frac{K_3}{m_3^2}\Big]\,m_1^2\,m_2^2\,m_3^2\Big\} \\
&\qquad + \Big\{-\;\Big[K_1 m_1^3 \;+\; K_2 m_2^3 \;+\; K_3 m_3^3\Big]\,p^2 \;+\; \Big[\frac{K_1}{m_1} \;+\; \frac{K_2}{m_2} \;+\; \frac{K_3}{m_3}\Big]\,m_1^2\,m_2^2\,m_3^2\Big\}\Big] \\
&\times \Big[(p^2 - m_1^2)(p^2 - m_2^2)(p^2 - m_3^2)\Big]^{-1} \quad . \qquad (2.14)
\end{aligned}$$

Here K_1 has been retained so as to exhibit the symmetries of the expression.

This way we have asymptotically $\widetilde{G}_F \to |p|^{-3}$. Together with the asymptotic character $|p|^{-2}$ of the Boson propagator one sees that the elementary Feynman graphs for the Fermion self-energy, the Boson self-energy, and the vertex correction, have asymptotic behavior as $|p|^{-5}$, $|p|^{-6}$, and $|p|^{-8}$, respectively. The integrals $\int d^4p(\cdot)$ thus are absolutely convergent, and no auxiliary (ghost) particles are required for the Boson field. The choice (2.2), (2.3) for the g-functional thus turns out to be satisfactory. If one has chosen $c_2 < c_3$ then ψ_2 is a ghost field and $\xi_2 = -1$, $\xi_1 = \xi_3 = 1$ in (2.5).

The fields $\psi_i(x)$ have the usual momentum space expansion:

$$\psi_{i\alpha}(x) \;=\; \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_i}{\omega_i}} \;\Big[b_i(\mathbf{p})\; u_i^{(\alpha)}(\mathbf{p})\; e^{i(\mathbf{px}-\omega_i t)} \;+\; d_i^\dagger(\mathbf{p})\; v_i^{(\alpha)}(\mathbf{p})\; e^{-i(\mathbf{px}-\omega_i t)}\Big] \tag{2.15}$$

where $\omega_i = +\sqrt{\mathbf{p}^2 + m_i^2}$, and the operators b, d obey the usual, ξ- and ϵ-modified anti-commutation relations

$$\Big[b_i^{(\alpha)}(\mathbf{p}),\; b_j^{(\beta)\dagger}(\mathbf{q})\Big]_\epsilon \;=\; \xi_i\,\delta_{ij}\,\delta_{\alpha\beta}\,\delta^3(\mathbf{p}-\mathbf{q}) \tag{2.16a}$$

$$\Big[d_i^{(\alpha)}(\mathbf{p}),\; d_j^{(\beta)\dagger}(\mathbf{q})\Big]_\epsilon \;=\; \xi_i\,\delta_{ij}\,\delta_{\alpha\beta}\,\delta^3(\mathbf{p}-\mathbf{q}) \quad , \tag{2.16b}$$

and where all other anti-commutators vanish.

One now can compute the renormalization constants, i.e., the mass shifts for the Fermion and the Boson, the vacuum-polarization shielding of the coupling constant, and the normalization constants, which all are finite, and proceed to compute the S-matrix. In view of the convergence of the renormalization constants here Källen's theorem [12] applies; thus according to that theorem the renormalized results are independent of the renormalization scheme, which means that all choices of the g-functionals having the same limiting form and yielding convergent Feynman graphs will give identical results.

We conclude this section by indicating the manner in which contact singularities arise in the Feynman – Schwinger integrals. To that end, consider the Fermion self-energy which is in second order

$$\Sigma \;=\; \text{Ø} \int d^4x\, d^4y \;\sum_{ijkl}\; \kappa_i\,\kappa_j\,\kappa_k\,\kappa_l\; \bar{\psi}_i(x)\,\gamma_5\,\Delta(x-y)\,S_{kl}(x-y)\,\gamma_5\,\psi_j(y) \quad . \tag{2.17}$$

Here Ø indicates that the integration has to be over an open domain so as not to include the contact terms [7, 8], and $\bar{\psi}_i(x)$ and $\psi_j(y)$ serve the role of test functions. The Wightman functions which retain the contact singularities are computed over the full domain and hence then Ø is to be omitted.

To evaluate this expression with the aim of localizing the most singular terms we recall

$$\Delta(x) \;=\; \frac{i}{4\pi^2}\,\left\{\frac{1}{x^2-i\varepsilon}-\frac{\mu^2}{4}\,\log\Big(-\,\tfrac{1}{4}\,\mu^2\,x^2\,\widetilde{\gamma}^2+i\varepsilon\Big)+\ldots\right\} \tag{2.18a}$$

$$\log(-z+i\varepsilon) \;=\; \log|z|+i\pi\,\theta(z) \tag{2.18b}$$

$$S_{ij}(x) \;=\; \delta_{ij}\,S_j(x)\,K_j \tag{2.19a}$$

$$\begin{aligned} S_j(x) \;=\;& \frac{i}{4\pi^2}\,\left\{\frac{2\gamma x}{(x^2-i\varepsilon)^2}+\frac{m_j}{2}\,\frac{1}{x^2-i\varepsilon}\right. \\ &-\;\left. m_j^2\,\left[\,\frac{\gamma x}{x^2-i\varepsilon}\;+\;\tfrac{1}{4}\,\log\Big(-\tfrac{1}{4}\,m_j^2\,x^2\,\widetilde{\gamma}^2+i\varepsilon\Big)\right]+\ldots\right\}\quad . \end{aligned} \tag{2.19b}$$

Here $\widetilde{\gamma}=1.781\ldots$. Inserting the expansions (2.18), (2.19b) in (2.17) we obtain, putting $y=0$ in view of translational invariance

$$\begin{aligned} \Sigma \;\cong\;& \text{Ø}\int d^4x\,\bar{\psi}(x)\,\gamma_5\sum_j K_j\,\left\{\frac{2\gamma x}{(x^2-i\varepsilon)^3}\;+\;\frac{m_j}{2}\,\frac{1}{(x^2-i\varepsilon)^2}\right. \\ &-\;m_j^2\,\frac{\gamma x}{(x^2-i\varepsilon)^2}\;-\;\frac{\mu^2}{4}\,\frac{\gamma x}{(x^2-i\varepsilon)^2}\,\log\Big(-\,\frac{1}{4}\,\mu^2\,x^2\,\widetilde{\gamma}^2+i\varepsilon\Big) \\ &-\;\left.\frac{m_j^2}{4}\,\frac{1}{x^2-i\varepsilon}\,\log\Big(-\,\frac{1}{4}\,m_j^2\,x^2\,\widetilde{\gamma}^2+i\varepsilon\Big)+\ldots\right\}\gamma_5\,\psi(0)\quad . \end{aligned} \tag{2.20}$$

From (2.13) and (2.15) we see that with the choice of the constants κ_i of (2.10) the first two terms of the "individual" propagators $S_j(x)$ cancel in the summation of (2.20) and do not appear in Σ. However, in view of our present task we shall look at one of the propagators, $S_j(x)$, leaving the cancellation to the end.

Following Ref. [7] we define $\bar{\chi}_\mu(x)$ such that there holds

$$\bar{\psi}(x) \;=\; x_\mu\,\bar{\chi}_\mu(x)+\bar{\psi}(0) \tag{2.21a}$$

$$\;=\; x_\mu\,\bar{\chi}_\mu(x)+\int d^4x'\,\bar{\psi}(x')\,\delta^4(x')\quad . \tag{2.21b}$$

Consider now the second term of the left-hand side of (2.20). Inserting (2.21a) in that term we have

$$\begin{aligned} I_j \;\cong\;& \frac{m_j}{2}\,\int d^4x\,\bar{\chi}_\mu(x)\,\frac{x_\mu}{(x^2-i\eta)^2}\,\psi(0) \\ &+\;\frac{m_j}{2}\int d^4x\,\left(\int d^4x'\,\bar{\psi}(x')\,\delta^4(x')\psi(0)\right)\,\frac{1}{(x^2-i\eta)^2} \end{aligned} \tag{2.22}$$

where we have used the version (2.21b) in order to make manifest the contact character of this term. The second term of (2.22) shows that I, and hence the individual terms of Σ, can contain contact singularities with amplitude proportional to

$$\int d^4x \; \frac{1}{(x^2 - i\eta)^2} \quad . \tag{2.23}$$

As shown in detail in Ref. [7], Eq. (2.23) indeed is logarithmically divergent. However, being a contact term it can contribute only to the Wightman function, and not to the S-matrix. At any rate, here it is eliminated upon summation over j in (2.20), also for the Wightman function. All remaining terms are regular for $x_\mu = 0$ and are best evaluated in momentum space. Their sum yields convergent integrals, individually for each j.

To summarize this Section, we have seen that in the g-functional formulation all matrix elements of the S-matrix are given by convergent expressions and hence are mathematically rigorously defined, and the Green's functions are those associated with the Lagrangian. In this the present formulation differs from the conventional regularization procedure where the regularized Green's function is disconnected from the Lagrangian and where the choice of the regularizing function is arbitrary and only dictated by convenience. At any rate, in view of Källen's theorem [12] the results of the conventional procedure turn out to be "correct" in the sense that they agree with those obtained as rigorous solutions of the g-functional formulation.

3 QUANTUM ELECTRODYNAMICS

As another example we now consider QED. Thus we have to take care of charge conservation, in addition to Lorentz invariance. We take here the g-functional

$$L\{\psi \cdots\} \;=\; L_F(\lambda) \;+\; L_B(\lambda) \;+\; L_I(\lambda) \tag{3.1}$$

with

$$L_F(\lambda) \;=\; -\int \; d^4x \;:\; [\bar\psi_1(x)(\gamma\partial + m_1)\psi(x) \;-\; \bar\psi_2(x)(\gamma\partial + m_2)\psi_2(x)] \;: \tag{3.2}$$

$$L_B \;=\; -\int \; d^4x \;:\; \left[\frac{1}{4}\; F_{\mu\nu}(x)F_{\mu\nu}(x) \;-\; \sum_\nu \; \Big(\partial_\mu B_\nu \partial_\mu B_\nu \;-\; M^2 B_\nu B_\nu\Big)\right] \;: \tag{3.3}$$

$$L_I \;=\; -i4 \int \; d^4x \;:\; [(e_1\bar\psi_1\gamma_\mu\psi_1) \;+\; (e_2\bar\psi_2\gamma_\mu\psi_2)] \;\; (A_\mu + B_\mu) \;:\quad . \tag{3.4}$$

and

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{3.5}$$

This g-functional thus contains besides the Fermion and the photon field two ghost fields: one Fermion and one Boson. The latter is a mixture of spin 0 and spin 1.

We check for charge conservation. To that end we introduce the substitution

$$\psi_1 \;\rightarrow\; e^{i\alpha}\,\psi_1 \qquad , \qquad \psi_2 \;\rightarrow\; e^{i\beta}\,\psi_2 \quad , \tag{3.6}$$

with α, β infinitesimal, and we demand

$$\frac{\delta L}{\delta \alpha} \;=\; 0 \qquad , \qquad \frac{\delta L}{\delta \beta} \;=\; 0 \quad . \tag{3.7}$$

From Noether's theorem we have

$$\partial_\mu\, j_\mu^{(1)} \;=\; 0 \qquad , \qquad \partial_\mu\, j_\mu^{(2)} \;=\; 0 \quad , \tag{3.8}$$

$$j_\mu^{(n)} \;=\; i\, e_n\, \bar{\psi}_n\, \gamma_\mu\, \psi_n \quad , \tag{3.9}$$

i.e., separately conserved currents. With the ansatz Eq. (3.4) here, in contrast to the Pauli-Villars procedure, no photo-excitation from ψ_1 to ψ_2 exists. Therefore one is free to chose the ghost coupling constant, e_2 at will. Thus in the Fermion mass-renormalization graph asymptotically at large loop momenta the Fermion propagator retains the character $\mid p \mid^{-1}$.

The only place where the Fermion ghost contributes is in the vacuum polarization, in that for every closed Fermion particle loop there exists an analogous ghost loop. The lowest order contribution then is

$$\Pi \;=\; \emptyset \int \left[\frac{e_1^2}{(\not{p} - m_1)(\not{q} - m_1)} \;-\; \frac{e_2^2}{(\not{p} - m_2)(\not{q} - m_2)} \right] d^4p \tag{3.10}$$

where

$$q \;=\; p - k \tag{3.11}$$

and where the minus sign of the ghost contribution results from the ghost field having the "wrong" statistics, i.e., $\epsilon = -1$ in (2.5). In each of these terms the quadratic divergence is a contact singularity and does not contribute to the open integral, while the surviving terms are

$$e_1^2\, \emptyset \int\, d^4p\, \frac{p^2 q^2 (m_1^2 - \zeta^2 m_2^2) \;+\; \cdots}{(p^2 - m_1^2)(q^2 - m_1^2)(p^2 - m_2^2)(q^2 - m_2^2)} \quad . \tag{3.12}$$

Choosing

$$\zeta^2 \;=\; \frac{e_2^2}{e_1^2} \;=\; \frac{m_1^2}{m_2^2} \tag{3.13}$$

the integral is absolutely convergent. With this choice in the limit $m_2^2 \;\rightarrow\; \infty$ the Fermion ghost not only freezes out but also decouples.

We now turn to the Boson sector. The photon propagator is the usual one (Feynman gauge)

$$\Delta_{ph} \;=\; \int \frac{\delta_{\mu\nu}}{p^2 - i\eta}\, d^4p \tag{3.14}$$

while for the ghost it is directly the Klein-Gordon propagator

$$\Delta_G \;=\; \int \frac{\delta_{\mu\nu}}{p^2 + M^2 - i\eta}\, d^4p \quad . \tag{3.15}$$

The full (photon + ghost) Boson propagator thus is

$$\Delta \;=\; -\int \frac{\delta_{\mu\nu}\; M^2}{p^2(p^2 - M^2) \;-\; i\eta}\, d^4p \quad ; \tag{3.16}$$

this way together with the Fermion propagator $(\not{p} - m)^{-1}$, all graphs are absolutely convergent for $|p| \to \infty$, and the limits $m_2^2,\ M^2 \to \infty$ can be safely carried out.

4 FORM OF THE STATIONARY STATE SOLUTIONS

Having seen that the S-matrix can be computed in a mathematically solid manner, we now shall show that the solutions themselves are well defined and can be used to construct the wave functions of the system, which then can be used to evaluate its characteristics exactly as in the familiar non-relativistic quantum mechanics. It is here that the modification (i) mentioned in the Introduction becomes explicitly evident.

The computation of the S-matrix in the interaction picture is based on a time-dependent treatment. Of course, this treatment can be applied to the computation of stationary states [20, 21, 22, 23]. On the other hand a stationary state is easiest to visualize in the time-independent treatment. Therefore, we will begin by illustrating the form of the solution in the Schrödinger picture using as examples the case of a single (physical) particle of the Yukawa theory given above, and that of a composite bound state system for the case of different Fermions, for example the bound state of a muon and an electron. We then will re-state the problem in the interaction picture and derive a covariant eigenvalue equation in terms of the U-matrix elements. Of course, for a convergent (or, within its range of convergence, for a semi-convergent) theory the results are independent of the choice of the method. In particular, it is well known that the Schrödinger-picture treatment is fully equivalent to the treatment by the Feynman graph expansion, only being enormously more cumbersome for performing an actual calculation.

In the Schrödinger picture the state vector of a given system, $|W^{(S)}(t)\rangle$, obeys the equation

$$\frac{\partial}{\partial t}\, |W^{(S)}(t)\rangle \;=\; -\,i\; H(\lambda)\; |W^{(S)}(t)\rangle \tag{4.1}$$

where

$$H(\lambda) \;=\; H_0(\lambda) + H_1(\lambda) \tag{4.2}$$

is the Hamiltonian of the system; $H_0(\lambda)$ and $H_1(\lambda)$ are associated with $L_0(\lambda) \;=\; L_F(\lambda) + L_B(\lambda)$ and $L_I(\lambda)$, of Eqs (2.1) through (2.4), respectively. The basic fields are those of the Heisenberg picture taken at the time $t = 0$. They are taken to be solutions of $H_0(\lambda)$. So, for example, the Fourier decomposition of the Schrödinger-picture fields $\psi_i^{(S)}(x)$ is

$$\begin{aligned} \psi_i^{(S)}(x) \;&=\; \psi_i^{(+)(S)}(x) + \psi_i^{(-)(S)}(x) \\ &=\; \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_i}{\omega_i}} \left(b_{\mathbf{p}}\, u_{\mathbf{p}}\, e^{i\mathbf{px}} \;+\; d^\dagger_{\mathbf{p}}\, v_{\mathbf{p}}\, e^{-i\mathbf{px}} \right) \end{aligned} \tag{4.3}$$

where we have suppressed the spin indices, and similarly for the other basic fields of the Lagrangian. Also, we use the notation $\omega_i \;=\; +\sqrt{\mathbf{p}^2 + m_i^2}$. In (4.3) we have introduced the usual notation for the positive frequency ($\psi^{(+)}$, annihilation operator) and negative frequency ($\psi^{(-)}$, creation operator) fields. In principle the basis (4.3) must be discretized in some manner, e.g., by Weyl's eigendifferential method. From now on we drop the superscript (S) indicating the Schrödinger picture, and the explicit reference to the parameter λ.

We note the commutation relations

$$\left[b_{\alpha\mathbf{p}},\, b^\dagger_{\beta\mathbf{q}} \right]_\epsilon \;=\; \xi\delta_{\alpha\beta}\; \delta^3(\mathbf{p} - \mathbf{q}) \tag{4.4}$$

$$\left[d_{\alpha\mathbf{p}},\, d^\dagger_{\beta\mathbf{q}} \right]_\epsilon \;=\; \xi\delta_{\alpha\beta}\; \delta^3(\mathbf{p} - \mathbf{q}) \tag{4.5}$$

$$\left[a_{\alpha\mathbf{p}},\, a^\dagger_{\beta\mathbf{q}} \right]_- \;=\; \xi\delta_{\alpha\beta}\; \delta^3(\mathbf{p} - \mathbf{q}) \tag{4.6}$$

where as above $\xi \;=\; +1$ for particles and $=\; -1$ for ghosts, and $\epsilon \;=\; -1$ for the "wrong" statistics.

A solution for a stationary state of a given system can be obtained by expansion in the Fock-space representation, i.e., by constructing a complete basis of states having the quantum numbers of the desired state (momentum, spin, etc.) and diagonalizing the total Hamiltonian, Eq. (4.2), in that basis. The Schrödinger-picture state vector for the system consisting of a single Fermion of momentum $\mathbf{p}$ at infinite distance of any other particles then will have the general form (suppressing the spin indices)

$$|W(t)\rangle \;=\; e^{-iEt}\; |W\rangle \quad , \tag{4.7}$$

$$|W\rangle \;=\; C_{100}\; b^\dagger_{\mathbf{p}}\; |0\rangle \quad + \quad \int d^3k\; C_{101}(\mathbf{k})\; b^\dagger_{\mathbf{p}-\mathbf{k}}\; a^\dagger_{\mathbf{k}}\; |0\rangle$$

$$+ \quad \int d^3k_1\, d^3k_2\; C_{102}(\mathbf{k}_1\mathbf{k}_2)\; b^\dagger_{\mathbf{p}-\mathbf{k}_1}\; a^\dagger_{\mathbf{k}_1-\mathbf{k}_2}\; a^\dagger_{\mathbf{k}_2}\; |0\rangle + \ldots$$

$$+ \quad \int d^3p_1\, d^3p_2\; C_{210}(\mathbf{p}_1\mathbf{p}_2)\; b^\dagger_{\mathbf{p}-\mathbf{p}_1}\; b^\dagger_{\mathbf{p}_1-\mathbf{p}_2}\; d^\dagger_{\mathbf{p}_2}\; |0\rangle + \ldots$$

$$+ \quad \ldots \tag{4.8}$$

where actually, e.g., $b^\dagger_{\mathbf{k}} = \sum_i b^\dagger_{i\mathbf{k}}$ to account for the auxiliary fields as implied by (2.2). In (4.7), E is the energy, i.e., $E = +\sqrt{\mathbf{p}^2 + M^2}$, where M is the at this juncture unknown mass of the system.

For the composite system we assume two distinct Fermions but only one Boson species, and their assorted auxiliary particles. The state vector for the composite system is similarly to above

$$\begin{aligned} |W^{(c)}\rangle \;\; = \;\; & \int d^3p_1 C_{11000}(\mathbf{p}_1)\; b^{(a)\dagger}_{\mathbf{p}-\mathbf{p}_1} b^{(b)\dagger}_{\mathbf{p}_1}\; |0\rangle \\ & + \; \int d^3p_1\, d^3k\; C_{11001}(\mathbf{p}_1\mathbf{k})\; b^{(a)\dagger}_{\mathbf{p}-\mathbf{p}_1}\; b^{(b)\dagger}_{\mathbf{p}_1-\mathbf{k}}\; a^\dagger_{\mathbf{k}}\; |0\rangle + \ldots \\ & + \; \int d^3p_1\, d^3p_2 d^3p_3\; C_{22110}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)\; b^{(a)\dagger}_{\mathbf{p}-\mathbf{p}_1}\; b^{(b)\dagger}_{\mathbf{p}_1-\mathbf{p}_2}\; b^{(a)\dagger}_{\mathbf{p}_2-\mathbf{p}_3}\; d^{(a)\dagger}_{\mathbf{p}_3}\; |0\rangle + \ldots \\ & + \; \ldots \end{aligned} \tag{4.9}$$

In the usual manner we now compute the Hamiltonian matrix by inserting (4.8) in (4.1) and by multiplying on the left with the Hermitian conjugate of one of the components of the state vector at a time, for example, $\langle\{\beta\}| = \langle 0|\; b_{\mathbf{p}-\mathbf{k}}\, a_{\mathbf{k}}$. This way we obtain the matrix equation

$$(E - E_{\{\beta\}})\; C_{\{\beta\}} \;\; = \;\; \sum_{\{\alpha\}} \; \int d^3k_1 \;\; \ldots \;\; C_{\{\alpha\}} \langle\{\beta\}|\; H_1\; |\{\alpha\}\rangle \tag{4.10}$$

where the fact has been used that H_0 is diagonal in the Fock space configurations, yielding the value $E_{\{\beta\}}$ for the configuration $\{\beta\}$.

The matrix (4.10) now can be diagonalized, either directly, i.e., non-perturbatively, or perturbatively, say, by the Rayleigh – Schrödinger method, as a power series in the coupling constant, say g, i.e., by writing

$$E \;\; = \;\; E_0 + gE_1 + g^2E_2 + g^3E_3 + \ldots \quad , \tag{4.11}$$

where for the case of semi-convergence the series has to be terminated. (Equivalently, in the non-perturbative treatment the Hamiltonian matrix has to be truncated.) One has, for example,

$$g^2\, E_2 \;\; = \;\; \sum \{\alpha\} \int d\{\alpha\}\; \frac{\langle\{1\}|\; H_1\; |\{\alpha\}\rangle\; \langle\{\alpha\}|\; H_1\; |\{1\}\rangle}{E_\alpha - E} \tag{4.12}$$

where $\{1\}$ is the ground configuration, i.e., the first term of (4.8). The integration in (4.12) is over the quantum numbers of the configuration $\{\alpha\}$, for example, the momentum $\mathbf{k}$ for the configuration $\{\alpha\} = b^\dagger_{\mathbf{p}-\mathbf{k}}\, a^\dagger_{\mathbf{k}}$:

$$\int d\{\alpha\}\ \frac{\langle\{b^\dagger_{\mathbf{p}}\}|\ H_1\ |\{\alpha\}\rangle\ \langle\{\alpha\}|\ H_1\ |\{b^\dagger_{\mathbf{p}}\}\rangle}{E_\alpha - E} \;=\; \int d^3k\ \frac{\langle\{b^\dagger_{\mathbf{p}}\}|\, H_1\, |b^\dagger_{\mathbf{p}-\mathbf{k}}\, a^\dagger_{\mathbf{k}}\rangle\, \langle b^\dagger_{\mathbf{p}-\mathbf{k}}\, a^\dagger_{\mathbf{k}}|H_1|\{b^\dagger_{\mathbf{p}}\}\rangle}{\omega_{\mathbf{k}} + \omega_{\mathbf{p}-\mathbf{k}} - E}\quad . \tag{4.13}$$

In the conventional formulation integrals of this type diverge. Owing to the auxiliary particles they are non-divergent in the g-functional formulation; thus Fock space is well-defined. As long as the configurations $\{\alpha\}$, $\{\beta\}$, retained in the Hamiltonian matrix belong to those appearing as intermediate states in the Feynman graphs connected with the terms of the convergent part of the presumably semi-convergent series expansion for the S-matrix, they have physical significance. If they belong to that part of the series expansion where the terms have started to grow, their retention would tend to diminish the accuracy of the results; and thus they should be omitted. This condition thus represents the criterion for the truncation of the secular equation, Eq. (4.10), needed for the direct diagonalization. (Further details of the solution of field theory problems by diagonalization in the Schrödinger picture, in particular the treatment of the relativistic center-of-mass, have been presented in Refs. [24, 25].)

Since the Feynman integrals converge and themselves are independent of g^2, E_2 is independent of g^2, and E_n is independent of g^n. Hence, the eigenvalue E of the solution joins smoothly to the non-interacting theory. More precisely: if in fact the S-matrix expansion is only semi-convergent, then as g^2 becomes smaller the beginning of the growth of the terms in the expansion gets postponed, and for $g^2 \to 0$ this beginning occurs at arbitrarily large order n. Thus, as mentioned in point (j) in the Introduction, E, Eq. (4.11), and also the S-matrix itself, in the g-functional formulation is analytic at the point $g^2 = 0$.

These analytic properties of the point $g^2 = 0$ are seen by the following limiting procedure: Given a value of λ, and a value of N, then in

$$E^{N,\lambda}(g) \;=\; \sum^{N}\ g^n\ E^\lambda_n \tag{4.14}$$

there exists a $g_0 \neq 0$ such that for $|g| \leq |g_0|$ and $n \leq N$ there holds

$$n!\ g^n\ E^\lambda_n \;\leq\; (n-k)!\ g^{n-k}\ E^\lambda_{n-k} \tag{4.15}$$

for positive integer k. Thus for $g \to 0$ the value of $E^{N,\lambda}(g)$ and its n-th derivatives with $n \leq N$ are independent of N.

We note the form of the wave function of the system. It is given by [re-introducing the

superscript (S)]

$$\eta(x_i,t) \;=\; \langle 0|\; \Psi^{(S)}\; |W^{(S)}(t)\rangle \tag{4.16}$$

where for the single Fermion case $\Psi^{(S)}$ is the field

$$\Psi^{(S)} \;=\; \sum_{\{\kappa\}} \prod_{i}^{i_\kappa} \bar{\psi}^{(+)}(\mathbf{x}_i) \prod_{j}^{j_\kappa} \psi^{(+)}(\mathbf{x}_j) \prod_{k}^{k_\kappa} \varphi^{(+)}(\mathbf{x}_k) \quad . \tag{4.17}$$

For the composite system the field is

$$\Psi^{(S)} \;=\; \sum_{\{\kappa\}} \prod_{i}^{i_\kappa} \bar{\psi}^{(a)(+)}(\mathbf{x}_i) \prod_{j}^{j_\kappa} \psi^{(a)(+)}(\mathbf{x}_j) \prod_{k}^{i_\kappa} \bar{\psi}^{(b)(+)}(\mathbf{x}_k) \prod_{l}^{l_\kappa} \psi^{(b)(+)}(\mathbf{x}_l) \prod_{m}^{m_\kappa} \varphi^{(+)}(\mathbf{x}_m) \quad . \tag{4.18}$$

For the one-Fermion system only the following terms of the full set of configurations, $\{\kappa\}$, are needed

$$\begin{aligned} \Psi^{(S)}_{(1)} \;=\;& \psi^{(+)}(\mathbf{x}_1) \left[1 + \varphi^{(+)}(\mathbf{y}_1) + \varphi^{(+)}(\mathbf{y}_1)\varphi^{(+)}(\mathbf{y}_2) + \ldots\right] \\ &+\; \bar{\psi}^{(+)}(\mathbf{x}_1)\; \psi^{(+)}(\mathbf{x}_2)\; \psi^{(+)}(\mathbf{x}_3) \left[1 + \varphi^{(+)}(\mathbf{y}_1) + \varphi^{(+)}(\mathbf{y}_1)\; \varphi^{(+)}(\mathbf{y}_2) + \ldots\right] \\ &+\; \ldots \quad . \end{aligned} \tag{4.19}$$

Analogously only a subset of the configurations of Eq. (4.18) is needed for the composite system.

According to (4.7) and (4.8) the wave function $\eta(x_i,t)$ of (4.16) has the overall time dependence e^{-iEt} and is a mixture of configurations containing different numbers of quanta of the basic fields. This is fully analogous to the configuration mixing in atomic or nuclear physics. One may call $\Psi^{(S)}$ of (4.17) the general Schrödinger-picture solution field, and $|W^{(S)}(t)\rangle$, (4.8), the Schrödinger-picture solution state vector for the one-Fermion system. And, given that the basic fields obey local commutation relations it is obvious that this is not the case for the solution fields (4.17) or (4.19). As can easily be verified, the commutation relation $\left[\Psi^{(S)}, \bar{\Psi}^{(S)}\right]_+$ – which is the equal-time relation of the corresponding interaction picture fields – is a mixture of c-numbers, creation operators, annihilation operators, and number operators, and is non-local; i.e., these fields do not commute at space-like separation, which is contrary to one of the axioms of constructive field theory. This way one of the basic assumptions of Haag's theorem,[1, 2, 3] i.e., that the equal-time commutation relations of the fields Ψ yield a local c-number, is not fulfilled; hence, that theorem does not apply in our g-functional formulation of QFT, and hence the interaction picture exists for non-trivial theories.

As the last point we now indicate that the wavefunction can be computed also in a covariant manner. To achieve this we now have to re-express (4.1) in terms of the U-matrix. This will

also show the connections with the results of the previous Sections. To that end we re-write our expressions in the interaction picture. We define for the state vector of the system in the stationary state n of energy E_n

$$|W_n^{(I)}(t)\rangle \;=\; e^{-iE_nt}\;|W_n^{(I)}\rangle \;=\; \sum_{\{\alpha\}}\; w_{\{\alpha\}}^{(n)(I)}(t)\;B_{\{\alpha\}}^{\dagger}|0\rangle \quad . \tag{4.20}$$

Here we have compactified the notation of (4.8) by introducing the combined index $\{\alpha\}$. The sum over $\{\alpha\}$ in (4.20) thus contains both the summation over the configurations and the integration (actually summation for a Weyl representation) over the momenta. The product of the creation operators of (4.8) is denoted by $B_{\{\alpha\}}^{\dagger}$. Similarly, we re-write eq (4.17) for the interaction-picture fields as

$$\Psi^{(I)}(t) \;=\; \sum_{\{\alpha\}}\;\prod_{abc}^{\{\alpha\}}\;\bar{\psi}_a^{(+)}(t)\;\psi_b^{(+)}(t)\;\varphi_c^{(+)}(t) \;\equiv\; \sum_{\{\alpha\}}\;\Psi_{\{\alpha\}}^{(+)(I)}(x_i,t) \tag{4.21}$$

taken at the same time t for all fields. In (4.21) the indices, a, b, c, denote also the momenta in the field expansions according to (4.3), for example.

Herewith, we have

$$\eta_n(x_i,t) \;=\; \langle 0|\;\Psi^{(+)(I)}\;|W_n^{(I)}\rangle \tag{4.22}$$

Being in the interaction picture the time-dependence of the products of fields is

$$\Psi_{\{\alpha\}}^{(+)(I)}(x,t) \;=\; e^{-i\,E_{\{\alpha\}}t}\,\Psi_{\{\alpha\}}^{(+)(I)}(x_i) \tag{4.23a}$$

$$E_{\{\alpha\}} \;=\; \sum_{i}^{\{\alpha\}}\;E_i \tag{4.23b}$$

where the E_i are the energies of the individual fields participating in the configuration $\{\alpha\}$, eq (4.21). Hence, the amplitudes of the state vector for our state must have the time-dependence

$$w_{\{\alpha\}}^{(n)(I)}(t) \;=\; e^{-i(E_n-E_{\{\alpha\}})t}\;w_{\{\alpha\}}^{(n)(I)} \tag{4.24}$$

in order to achieve the overall time dependence of (4.20).

We now follow the development of the state beginning at time t_0 by means of the evolution operator U. Dropping the superscript (I) we have

$$|W_n(t_1)\rangle \;=\; U(t_1,t_0)\;|W_n(t_0)\rangle \tag{4.25}$$

or, in full detail

$$w_{\{\alpha\}}^{(n)}(t_1) \;=\; \sum_{\beta}\;U_{\alpha\beta}(t_1,t_0)w_{\{\beta\}}^{(n)}(t_0) \quad . \tag{4.26}$$

Combining (4.26) with (4.24) we obtain

$$e^{-i(E_n - E_{\{\alpha\}})(t_1 - t_0)}\ w^{(n)}_{\{\alpha\}} \ = \ \sum_\beta U_{\alpha\beta}(t_1, t_0)\, w^{(n)}_{\{\beta\}} \quad , \tag{4.27}$$

dropping the argument t_0 in the amplitudes. We now introduce the operator

$$e^{-i\int_{t_0}^{t_1}(E_n - H_0)dt'} \ = \ U^{(0)}(E_n; t_1, t_0) \tag{4.28}$$

which is diagonal in our basis. It has the structure of a mass counter term. Herewith, (4.26) can be written as

$$\sum_n \ \Big[\delta_{\alpha\beta}\, U^{(0)}_{\alpha\beta}(E_n; t_1 - t_0) - U_{\alpha\beta}(t_1 - t_0)\Big] \ w^{(n)}_{\{\alpha\}} \ = \ 0 \tag{4.29}$$

which is a (non-linear) eigenvalue equation for the energy and the state vector amplitudes. Upon the limit $t_0 \to -\infty$, $t_1 \to +\infty$, one arrives at the covariant expressions for the S-matrix elements which can be computed by the familiar Feynman graph expansion. This way we see that the concept of the wavefunction of a system is not limited to non-relativistic physics but has a precise meaning also in the frame of relativistic g-functional QFT.

5 SUMMARY

In this paper we have demonstrated that in the g-functional formulation of quantum field theory every step leading from the Lagrangian to the S-matrix can be defined unambiguously in a mathematically rigorous way; the renormalizations are non-divergent and can be carried through explicitly; and the state vectors are normalizable. As mentioned in point (E) of the Introduction, concerning the convergence properties of QFT we assume semi-convergence of the perturbation expansion. The solution fields are non-local and hence Haag's theorem does not apply; i.e., the interaction picture exists. The state vectors of the solutions are linear superpositions of Fock space configurations. This is true both for single particles, e.g., electrons, and for composite systems, e.g., positronium. For a small coupling constant, in the solution the ground configuration has the largest amplitude; the higher configurations are corrections. The solutions thus connect smoothly with the "free" theory; i.e., for a fixed λ the limit $g^2 \to 0$ within the limitations imposed by the possible semi-convergence of the expansion is analytic in the usual sense: for given ϵ and n, there is a g_0^2 such that for $g^2 < g_0^2$ the n^{th} configuration has an admixture $|A_n|^2 < \epsilon$. Herewith we have shown that non-trivial relativistic QFT is not only consistent mathematically, but also makes sense from the point of view of physics. Thus, the theory is "transparent," i.e., from contemplating the solutions one can draw conclusions concerning the underlying Lagrangian. This connection is not colored by the mathematics: the result does not depend on the choice of the g-functional, as long as it has the desired Lagrangian as the limiting form, and it has

the needed convergence properties. Conversely, one then is on safe grounds to judge whether a chosen Lagrangian, i.e., always defined as the limit of a g-functional, is suitable for describing the physics at hand. Thus we can conclude that non-trivial relativistic quantum field theory exists as a rigorous mathematical theory and that it can be employed in attempting the description of Nature.

APPENDIX A

g-functions and meta-g-functions

Generalized functions (g-functions), also called distributions, are defined in a limit procedure by their action on a function, the "test function." In the terminology of Lighthill [26] the limit procedure is carried out in terms of sets of "good functions" which contain a parameter, λ, and the g-functions arise in the limit $\lambda \to 0$. Considering as an example the $\delta-$ function, we have

$$\delta(x) \;=\; \lim_{\lambda\to 0} \Delta_\lambda(x) \tag{A.1}$$

and

$$\lim_{\lambda\to 0} \int \Delta_\lambda(x)\, f(x)\, dx \;=\; f(0) \quad . \tag{A.2}$$

Of course, (A.1) is only a symbolic notation and to be understood in the context of (A.2). There exist unlimitedly many different choices for the sets of "good functions" $\Delta_\lambda(x)$ which can be used. But all of them make mathematical sense only for $\lambda \neq 0$. The functions $\Delta_\lambda(x)$ are meaningless for $\lambda = 0$. The limit procedure thus is to be carried over the domain $[\lambda_0, 0)$ for λ, i.e., the domain open at $\lambda = 0$, and where λ_0 is some arbitrary, "small" value. Also, given a set $\Delta_\lambda(x)$, the functions $f(x)$, the "test functions," must have certain characteristics for (A.2) to hold.

All this is well-known; the mathematics of g-functions is essentially complete [26, 27]. The most essential of their characteristics is that their action, analogous to (A.2), is unambiguous, and that they have unambiguous Fourier transformations. Thus the Fourier transform of (A.2) is given by the convolution of the Fourier transforms of $\Delta_\lambda(x)$ and of $f(x)$.

When attempting to apply (A.2) to the case

$$f(x) \;=\; \frac{1}{x}\, g(x) \simeq \left(\frac{P}{x} \;+\; Z\,\delta(x)\right) g(x) \tag{A.3}$$

[$g(x)$ a "good" function] one arrives formally at

$$\lim_{\lambda\to 0} \int \Delta_\lambda(x)\, \frac{1}{x}\, g(x)\, dx \;=\; \lim_{\lambda\to 0} \int \Delta_\lambda(x) \left(\frac{P}{x} \;+\; Z\,\delta(x)\right) g(x)\, dx \tag{A.4}$$

which breaks the frame of g-functions; it involves the product of two g-functions. Such quantities have been called "meta-g-functions." [7] The Wightman functions of QFT frequently contain meta-g-functions.

Going back to (A.1) one finds

$$[\delta(x)]^2 \;=\; \lim_{\substack{\lambda_1\to 0\\ \lambda_2\to 0}} \Delta_{\lambda_1}(x)\,\Delta_{\lambda_2}(x) \quad . \tag{A.5}$$

One has here two independent limits. The result one would obtain from

$$\lim_{\lambda_1,\lambda_2\to 0}\int \Delta_{\lambda_1}(x)\,\Delta_{\lambda_2}(x)\,g(x)\,dx \tag{A.6}$$

depends on the manner one performs the limit procedure. Thus (A.6) is inherently ambiguous. Meta-g-functions unavoidably yield ambiguous results. Not much is known about their properties. Still, one can define their "singularity character," which is independent of the way one performs the limits, e.g., in (A.6). Thus one may call a meta-g-function $D^{(n)}(x)$ to be of degree n (n =integer) if there holds

$$\int\; D^{(n)}(x)\; x^{n+k}\,dx \;=\; 0 \tag{A.7}$$

with integer $k \geq 0$. With (A.7) one sees that the general solution of the equation (measure dx)

$$x^n\; f(x) \;=\; g(x) \tag{A.8}$$

in addition to g-functions ($\delta-$functions and their derivatives) contains also meta-g-functions of degree up to n. (The Eq. (A.8) can be used to define meta-g-functions of degree n [7]]. An example of a degree 2 meta-g-function is Eq. (A.6). Meta-g-functions of degree $n \;=\; 1$ are g-functions.) Some further discussion on the properties of meta-g-functions is contained in Ref. [7].

Contact type g-functions and meta-g-functions.

The open integral, $\emptyset\int_a^b dx\,\cdot$, is defined to leave out the single point $x = 0$. Precisely, it is defined as a definite integral over the range $\emptyset(a,b) = (a,0),(0,b)$. It differs from the integral $\int_a^b dx\,\cdot$ which has the range $(a,b) = (a,0]\,(0,b) = (a,0)\,[0,b)$. In order for these two integrals to be different, the integrand must be a (meta-)g-function of the contact type, for example, $\delta(x)$, or $[\delta(x)]^2$, etc. Thus, for example,

$$\emptyset\int dx\,[f(x)\,\delta(x) + g(x)] = \int dx\,g(x) \tag{A.9}$$

for $f(x)$ and $g(x)$ (ordinary) test-functions. The g-function $\delta'(x)$ is not of the contact type as can be seen from the definition

$$\delta'(x) \;=\; \lim_{\eta\to 0}\; \frac{\delta(x+\eta) - \delta(x-\eta)}{2\eta} \quad . \tag{A.10}$$

It is retained in the open integral. Further details are given in Ref. [7].

We could carry out our analysis of the meta-g-functions because we used the procedure of Lighthill [26]. The Schwartz – Gel'fand definitions [27] do not lend themselves to the required generalizations.

g-functionals and meta-g-functionals

In analogy to the g-functions one can define generalized functionals, for short g-functionals. They are functionals of the fields $\psi(x)$, which can be c-numbers or operators. In analogy to the good functions generating the g-functions, the functionals generating the g-functionals contain a parameter λ, and they are well-defined for $\lambda \neq 0$; that means that the functional equations they are associated with (e.g., the equations of motion) have solutions for ψ (and for the state vectors upon which the field operators act) for $\lambda \neq 0$. They make no mathematical sense for $\lambda = 0$. The g-functionals are then defined in analogy with the g-functions: in order to achieve mathematically sound results the calculation is performed for a non-zero, fixed value of λ. Only after the completion of the calculation one goes to the limit $\lambda \to 0$, in the strict sense that $\lambda = 0$ is not permitted; the limit procedure is to be carried out over the domain $[\lambda_0, 0)$, which is open at $\lambda = 0$, with arbitrary "small" λ_0. To emphasize: the limit procedure is carried out for the *solutions* of the theory (e.g., the S-matrix), i.e., the solutions obtained for "small" λ.

Furthermore, in complete analogy to g-function theory one can give for typical, general cases of g-functionals rules, for example, Feynman rules, on how to write the results of the $\lambda \to 0$ limit procedure. And, in the same way as for g-functions, one always can check the rules by actually computing the $\lambda \to 0$ results.

As is well-known, the solutions of the g-functional theories contain g-functions. For example, the Feynman integrals, say for QED, contain light-cone $\delta-$functions and their derivatives. These g-functions survive the $\lambda \to 0$ limit procedure. One may say that g-functionals lead to g-functions. Inasmuch as g-functions have well-defined characteristics, the solutions of g-functional theories are also fully, unambiguously defined.

One also can introduce the concept "meta-generalized-functionals" arising in the context of g-functionals as analogues of the meta-g-functions which arise in the context of g-functions. In the same way that meta-g-functions are inherently ambiguous, the meta-g-functionals are inherently ambiguous. Examples for the meta-g-functionals are the "non-renormalizable" quantum field theories: they cannot be made unambiguous – even by a $\lambda \to 0$ limit procedure. In fact, the Feynman integrals arising in non-renormalizable theories contain non-contact-type meta-g-functions, which belong in the solutions [7] and which render the solutions ambiguous. Again, one may say that

meta-g-functional theories lead to meta-g-functions.

APPENDIX B

Negative Metric Fields

Consider the action

$$L \;=\int \bar{\psi}(x)\,(\gamma\partial + m)\,\psi(x)\;d^4x \quad . \tag{B.1}$$

The Fourier expansion of the fields is

$$\psi(x) \;=\int \frac{d^3p}{(2\pi)^{3/2}}\,\sqrt{\frac{m}{E_p}}\;\left[b_{\mathbf{p}}\,u(\mathbf{p})\;e^{i(\mathbf{px}-Et)}\;+\;d^\dagger_{\mathbf{p}}\,v(\mathbf{p})\;e^{-i(\mathbf{px}-Et)}\right] \quad . \tag{B.2}$$

The Fock operators obey the usual anti-commutation relation except for the "wrong" sign:

$$\left[b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}\right]_+ \;=\; \left[d_{\mathbf{p}}, d^\dagger_{\mathbf{p}'}\right]_+ \;=\; -\,\delta^3(\mathbf{p}-\mathbf{p}') \quad . \tag{B.3}$$

The Hamiltonian here is

$$H \;=\; -\int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\;E_p\;\left\{:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:\;+\;:d^\dagger_{\mathbf{p}}d_{\mathbf{p}}:\right\} \quad . \tag{B.4}$$

The equations of motion are,

$$\begin{aligned}[H,\psi(x)]_- \;=\; -\int \frac{d^3p}{(2\pi)^{3/2}}\int \frac{d^3k}{(2\pi)^{3/2}}\;E_p\,\Big\{\Big[:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:b_{\mathbf{k}}\;-\;b_{\mathbf{k}}:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:\Big]\;u_{\mathbf{p}}\,e^{i(\mathbf{px}-E_pt)}\\ +\Big[:d^\dagger_{\mathbf{p}}d_{\mathbf{p}}:d^\dagger_{\mathbf{k}}\;-\;d^\dagger_{\mathbf{k}}:d^\dagger_{\mathbf{p}}d_{\mathbf{p}}:\Big]\;v_{\mathbf{p}}\,e^{-i(\mathbf{px}-E_pt)}\Big\} \quad .\end{aligned} \tag{B.5}$$

We have

$$\begin{aligned}:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:\;b_{\mathbf{k}}-b_{\mathbf{k}}\;:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:\;&=\;:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:\;b_{\mathbf{k}}-\left(-b_{\mathbf{k}}b^\dagger_{\mathbf{p}}-b^\dagger_{\mathbf{p}}b_{\mathbf{k}}+b^\dagger_{\mathbf{p}}b_{\mathbf{k}}\right)b_{\mathbf{p}}\\ &=\;:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:\;b_{\mathbf{k}}-\delta^3(\mathbf{k}-\mathbf{p})b_{\mathbf{k}}+b^\dagger_{\mathbf{p}}b_{\mathbf{k}}\;b_{\mathbf{p}}\\ &=\;:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:\;b_{\mathbf{k}}-\delta^3(k-p)b_{\mathbf{k}}-:b^\dagger_{\mathbf{p}}b_{\mathbf{p}}:b_{\mathbf{k}}\\ &=\;-\delta^3(\mathbf{k}-\mathbf{p})\;b_{\mathbf{k}}\end{aligned} \tag{B.6}$$

and similarly for the other terms. Thus

$$\begin{aligned}[H,\psi(x)]_- \;&=\; -\int \frac{d^3p}{(2\pi)^{3/2}}\;E_p\left\{b_{\mathbf{p}}\,u_{\mathbf{p}}\,e^{i(\mathbf{px}-E_pt)}-d^\dagger_{\mathbf{p}}\,v_{\mathbf{p}}\,e^{-i(\mathbf{px}-E_pt)}\right\}\\ &=\;\frac{1}{i}\,\frac{\partial}{\partial t}\,\psi(x)\end{aligned} \tag{B.7}$$

which agrees with the equations of motion for positive metric fields. The expressions for the interaction picture development thus remain unchanged.

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